

Using a quantum computer to investigate quantum chaos

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Abstract

We show that the quantum baker's map, a prototypical map invented for theoretical studies of quantum chaos, has a very simple realization in terms of quantum gates. Chaos in the quantum baker's map could be investigated experimentally on a quantum computer based on only 3 qubits.

Since the discovery that a quantum computer can in principle factor large integers in polynomial time [1,2], quantum information has become a major theoretical and experimental research topic, focusing on properties, applications, generation and preservation of highly entangled quantum states [3]. Although it is not clear if a full-scale quantum computer will ever be realized [4,5], experiments with quantum gates are being performed at present [6–9]. It is important to devise applications for early quantum computers which are incapable of doing large-scale computations like factoring.

Early quantum computers appear to be well suited to study the quantum dynamics of simple quantum maps. The quantum baker's map [10], one of the simplest quantum maps used in quantum chaos research, has been extensively studied in recent years [11–16]. Up to now, it has been regarded as a purely theoretical toy model. As a consequence of recent progress in the field of quantum computing [6–9], however, an experimental realization of the quantum baker's map seems possible in the very near future.

Any unitary operator can be approximated by a sequence of simple quantum gates [17–19]. The main result of this paper is that the quantum baker's map has a particularly simple realization in terms of quantum gates. The quantum baker's map displays behavior of fundamental interest even for a small dimension of Hilbert space. Numerical simulations [13] in $D = 16$ dimensional Hilbert space suggest that a rudimentary quantum computer based on as few as three qubits (i.e. three two-state systems spanning $D = 8$ dimensional Hilbert space) could be used to study chaos in the quantum baker's map. In particular, it may be possible to find experimental evidence for hypersensitivity to perturbation, a proposed information-theoretical characterization of quantum chaos [13,20–22].

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The classical baker's transformation [23] maps the unit square $0 \leq q, p \leq 1$ onto itself according to

$$(q, p) \mapsto \begin{cases} \left(2q, \frac{1}{2}p\right), & \text{if } 0 \leq q \leq \frac{1}{2}, \\ \left(2q - 1, \frac{1}{2}(p + 1)\right), & \text{if } \frac{1}{2} < q \leq 1. \end{cases} \quad (1)$$

This corresponds to compressing the unit square in the p direction and stretching it in the q direction, while preserving the area, then cutting it vertically, and finally stacking the right part on top of the left part—in analogy to the way a baker kneads dough.

To define the quantum baker's map [10], we quantize the unit square following [11,24]. To represent the unit square in D -dimensional Hilbert space, we start with unitary “displacement” operators \hat{U} and \hat{V} , which produce displacements in the “momentum” and “position” directions, respectively, and which obey the commutation relation [24]

$$\hat{U}\hat{V} = \hat{V}\hat{U}\epsilon, \quad (2)$$

where $\epsilon^D = 1$. We choose $\epsilon = e^{2\pi i/D}$. We further assume that $\hat{V}^D = \hat{U}^D = 1$, i.e., periodic boundary conditions. It follows [11,24] that the operators \hat{U} and \hat{V} can be written as

$$\hat{U} = e^{2\pi i \hat{q}} \quad \text{and} \quad \hat{V} = e^{-2\pi i \hat{p}}. \quad (3)$$

The “position” and “momentum” operators \hat{q} and \hat{p} both have eigenvalues j/D , $j = 0, \dots, D-1$.

In the following, we restrict the discussion to the case $D = 2^L$, i.e., the dimension of Hilbert space is a power of two. For consistency of units, let the quantum scale on “phase space” be $2\pi\hbar = 1/D = 2^{-L}$. A transformation between the position basis $\{|q_j\rangle\}$ and the momentum basis $\{|p_j\rangle\}$ is effected by the discrete Fourier transform F'_L , defined by the matrix elements

$$(F'_L)_{kj} = \langle p_k | q_j \rangle = \sqrt{2\pi\hbar} e^{-ip_k q_j / \hbar} = \frac{1}{\sqrt{D}} e^{-2\pi i k j / D}. \quad (4)$$

There is no unique way to quantize a classical map. Here we adopt the quantized baker's map introduced by Balazs and Voros [10] and defined by the matrix

$$T' = F'^{-1}_L \begin{pmatrix} F'_{L-1} & 0 \\ 0 & F'_{L-1} \end{pmatrix}, \quad (5)$$

where the matrix elements are to be understood relative to the position basis $\{|q_j\rangle\}$. Saraceno [11] has introduced a quantum baker's map with stronger symmetry properties by using antiperiodic boundary conditions, but in this article we restrict the discussion to periodic boundary conditions as used in [10].

The discrete Fourier transform used in the definition of the quantum baker's map (5) plays a crucial role in quantum computation and can be easily realized as a quantum network using simple quantum gates. The following discussion of the quantum Fourier transform closely follows [2]. The $D = 2^L$ dimensional Hilbert space modeling the unit square can be realized as the product space of L qubits (i.e. L two-state systems) in such a way that

$$|q_j\rangle = |j_{L-1}\rangle \otimes |j_{L-2}\rangle \otimes \cdots \otimes |j_0\rangle, \quad (6)$$

where $j = \sum j_k 2^k$, $j_k \in \{0, 1\}$ ($k = 0, \dots, L-1$), and where each qubit has basis states $|0\rangle$ and $|1\rangle$.

To construct the quantum Fourier transform, two basic unitary operations or *quantum gates* are needed: the gate A_m acting on the m th qubit and defined in the basis $\{|0\rangle, |1\rangle\}$ by the matrix

$$A_m = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad (7)$$

and the gate B_{mn} operating on the m th and n th qubits and defined by

$$B_{mn} |j_{L-1}\rangle \otimes \cdots \otimes |j_0\rangle = e^{i\phi_{mn}} |j_{L-1}\rangle \otimes \cdots \otimes |j_0\rangle, \quad (8)$$

where

$$\phi_{mn} = \begin{cases} \pi/2^{n-m} & \text{if } j_m = j_n = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (9)$$

In addition we define the gate S_{mn} which swaps the qubits m and n .

The discrete Fourier transform F_L can now be expressed in terms of the three types of gates as

$$\begin{aligned} F_L = S &\times (A_0 B_{01} \cdots B_{0,L-1}) \times \cdots \\ &\times (A_{L-3} B_{L-3,L-2} B_{L-3,L-1}) \\ &\times (A_{L-2} B_{L-2,L-1}) \times (A_{L-1}) \end{aligned} \quad (10)$$

where

$$S = \begin{cases} S_{0,L-1} S_{1,L-2} \cdots S_{L/2-1,L/2} & \text{for } L \text{ even,} \\ S_{0,L-1} S_{1,L-2} \cdots S_{(L-3)/2,(L+1)/2} & \text{for } L \text{ odd,} \end{cases} \quad (11)$$

reverses the order of the qubits. The quantum baker's map (5) is then given by

$$T = F_L^{-1} (I \otimes F_{L-1}), \quad (12)$$

where F_{L-1} acts on the $L-1$ least significant qubits, and I is the identity operator acting on the most significant qubit. The gates corresponding to the bit-reversal operator S can be saved if the qubits in the tensor product (6) are relabeled after each execution of F_L or F_{L-1} .

In $D = 8 = 2^3$ dimensional Hilbert space, one iteration of the quantum baker's map is performed by the short sequence of gates

$$T = S_{02} A_0 B_{01}^\dagger B_{02}^\dagger A_1 B_{12}^\dagger A_2 S_{01} A_0 B_{01} A_1. \quad (13)$$

This implementation of the quantum baker's map can be viewed in two complementary ways. On the one hand, it shows that the quantum baker's map can be efficiently simulated on a quantum computer. A 30-qubit quantum computer could perform simulations that are virtually impossible on present-day classical computers.

On the other hand, an iteration of the gate sequence (12) on an L -qubit quantum computer is a physical realization of the quantum baker's map. This opens up the possibility of an experimental investigation of chaos in a physical system in a purely quantum regime.

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